Combat Simulation:
Part (I) Overview of the Lanchester's Model

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Abstract

In the last years scientists have seen an increasing interest for the application of dynamic system theory to social science. This article is divided into two parts. In the first section we briefly review the so called Lanchester combat model. In particular we are going to examine the application of Lanchester model to the bloody battle of Iwo Jima. In the second part, we'll introduce non linear model and, in particular, we'll see the utility of bifurcation analysis. Employing bifurcation theory it's possible to understand the variations of the characteristic parameters that influence a state $u = u(\mathbf{x}, t)$ and how these parameters need to change to be, always, in advantage compared to the adversary.

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1 Introduction

Goals of scientific strategy are to recognize a threat, analyze and neutralize it. All the conflicts in 21^0 century, the so called digital era, are and will be characterized by an asymmetric threat. An important tool, that enable Commandants to analyze the threat and decide how to fight it, is simulation. Simulation means to replay the dynamics of a system and guarantee a certain degree of adherence[1] with reality. The Commandants, with a correct analysis of the simulation's output data, have at their disposal a decisional criterion. We recall simulation procedure. First it's necessary to create a mathematical model of the system which behavior we are interested in and then run the simulation. So it's an important subject combat modellization.
The relationships between mathematics and scientific strategy is very old. From very long time a great number of mathematicians employed their ingenious and their knowledge to carry out bellicose devices. For a throughly exposition see[2]. Since the beginnings of the 19^0 some mathematicians tried to understand the behavior of horrible events like battles by analytical tools. Important application of mathematic is to predict the behavior of some parameters that characterize a complex system formed, for example, by entity that moves on the fields and exchange, between them,informations. This is the case of an *adaptive complex system*, from now on ACS[2].

1.1 Dynamical System

To simulate an ACS we need of informations about its dynamic and complexity. The dynamic requires the concept of dynamical system that requires the knowledge of a graph [1]. A graph is described by vertex (point) and edge that connect vertexes. With the graph's concept at our disposal we can introduce the concept of space state. From an informal point of view we can say that a dynamical system is a structure that changes with time according to a precise rule and originates complex behavior. Every state of a dynamical system is represented by a vertex. If α, β are two points, it exists a function φ so that $\beta = \varphi(\alpha)$. Graph theory is quite simple and it's belong to the field of algebraic topology[3]. It's interesting observe that the name topology

[1] *...Certain degree of adherence..*means some precise that are out of the scope of the present article. It's the *validation procedure* and the interested reader could see the CCRP publication series

[2] adaptive means the complex system interacts with the ground

[3] *"Topology: The stratosphere of human thought! In the twenty-fourth century it might be of use of someone"...-The First Circle.*A.Solzhenitsksyn.

derives from ($\tau\grave{o}\pi o\zeta$) *topos*, that means place, and ($\lambda o\gamma o\zeta$) a variant of the verb ($\lambda\acute{\epsilon}\gamma\epsilon\iota\nu$) (*legein*) meaning to speak.
In the last years mathematicians have seen an increasing numbers of applications of Algebraic Topology, a very abstract field of higher mathematics, to many physical science. In particular *homology and cohomology* theories of Algebraic topology are useful to describe the the evolution of network structure[4]. At this point we recall the concept of dynamical system

Definition 1 (Dynamical System)
A dynamical system is given by the triple $\{T, X, \varphi\}$ *where* $T \subseteq \mathbf{R}$*,* X *is the state space,* $\varphi : X \longrightarrow X$ so that $\alpha \overset{\varphi}{\mapsto} \beta$

The Dynamical System Theory is a formalized structure that enable scientists to use suitable instruments necessary to the description and comprehension of a physical system. A continuous dynamical system is described by ODE [5]

$$\frac{\mathrm{d}\,u}{\mathrm{d}\,t} = \varphi\,(u, \omega) \qquad u \in \mathrm{X} \quad \alpha \in \mathbf{R}$$

ω is the so called control parameter, the function $\varphi \in C\,(\mathrm{X}, \mathbf{R}^n)$. This definition is a pillar of the so called dynamical system theory. For example see [4].[6]

1.2 Complexity

The definition of Complexity is more difficult[7]. From a mathematical point of view we observe that in the theory of graph there isn't a clear definition of complexity. In many topics of simulations the complexity measure is defined in terms of memory and time to run an algorithm[8]. The definition of the complexity, for a graph, is object of active research. If $\Gamma_{V,e}$ is a graph, $V_1, V_2, \cdots, V_n$ are the graph vertices and A is the adjacency matrix, formed by the elements:

$$g_{i,j} = \begin{cases} 1 & \text{if i is connected with j} \\ 0 & \text{if i is not connected with j} \end{cases} \tag{1}$$

[4]We have in mind the so-called Q-analysis by R.Atkin that at the beginnings of '70 years of the last century uses algebraic topology to describe social network

[5]Ordinary Differential Equations

[6]The Author discusses throughly the Lanchester theory

[7]In the article [7] there is an interesting approach to the complexity

[8]To be rigorous the *Kolmogorov-Chaitin* complexity $\Psi\,(x)$ of an objects x is the length, in bits, of the smallest program, in bits, that when run on a Universal Turing Machine outputs x and then halts

We associate to every $V_i \in \Gamma_{V,e}$ the linear form $f_i = \sum_{J=1}^{N} g_{i,j}\, x_j$. It's clear that f_i depends only on the neighbors of V_i. Different ordering of the vertexes of a graph give rise to possibly different matrices. The linear forms formed by adjacency matrix of a graph differs only by a permutation so it's possible to define the linear complexity $\mathrm{L}(\Gamma_{\mathrm{V,e}})$ of a graph. The definition is a measure of how hard is to compute $\mathrm{A} \cdot \mathrm{X}$, actually this equation tell us the memory needed by computer to multiply the adjacency matrix for a vector generate by the elements $g_{i,j}$.

2 Lanchester linear combat Model

During the first years of the 20° century analytical methods were used to describe horrible events like battles. In 1916 aeronautic engineer Frederick William Lanchester published the book *Aircraft in Warfare. The down of the Fourth Arm.* In this book the fight between two forces is described by a system of ordinary differential equations of the first degree with constant coefficients. This theory was developed, also, from the physician Lewys Frey Richardson. Lanchester model is described by a couple of differential equations.

$$\begin{cases} \frac{\mathrm{d}\, u}{\mathrm{d}\, t} = k\, u - \alpha\, v + g \\ \frac{\mathrm{d}\, v}{\mathrm{d}\, t} = l\, v - \beta\, v + h \end{cases}, \tag{2}$$

We indicate with $\mathrm{u}(\cdot)$ and $\mathrm{v}(\cdot)$ two time's functions that describe the power of the two entity that are fighting It's not useful to solve this system of ODE whilst is important the qualitative analysis of this system of equations. A qualitative study means the analysis of critical points. To this end we use Lyapunov theory that establishes the necessary and sufficient conditions for the stability of a system. These type of informations are useful in fact they are the suitable base to determine the behavior of the characteristic parameters. For a review see [4]. In matricial form we rewrite system (2) in the form

$$\mathbf{x}' = \mathrm{A}\, \mathbf{x} + \mathbf{g}\,(x)$$

where $\mathbf{g}\,(x)$ è is the vector

$$\mathbf{g}\,(x) = \begin{pmatrix} g \\ h \end{pmatrix}, \quad \text{while}\, \mathrm{A} = \begin{pmatrix} -\alpha & k \\ l & -\beta \end{pmatrix}$$

To perform the analysis of stability we study the characteristic polynomial $p\,(\lambda) = \det(\mathrm{A} - \lambda\, I)$. From this equation we get

$$p\,(\lambda) = \det \begin{pmatrix} -\alpha - \lambda & k \\ l & -\beta - \lambda \end{pmatrix}$$

The equilibrium points are:

$$N_1 = \frac{k\,h + \beta\,g}{\alpha\,\beta - k\,l}\,, N_2 \frac{l\,g + \alpha\,h}{\alpha\,\beta - k\,l}$$

For example if $\alpha\,\beta - k\,l < 0$ the equilibrium points is stable. The knowledge of this points is important because we can deduce the behavior in the plane $\{\mathrm{u}, \mathrm{v}\}$ and some statistic about the coefficients

2.1 Example: The Iwo Jima battle

We apply the Lanchester theory to the case of the bloody battle of Iwo Jima. This is a little volcanic island at 660 miles from Tokyo. The airfield of the island was an important strategic objective for the U.S. Army. In fact the fighters couldn't escort the air strikes, B-29 stratofortress, because the had a limitated endurance. This problem would be completely solved if the fighters could take off from Iwo Jima. Notwithstanding the war was loose for Japanese General Kuribayashi, chief of Japanese troops on the island, organized a fierce resistance . The Americans troops started the island invasion 19 February 1945, The battle ended 26 march 1945
In this case Lanchester model is

$$\begin{cases} \frac{\mathrm{d}\,u}{\mathrm{d}\,t} = -a\,v + P\,(t) \\ \frac{\mathrm{d}\,v}{\mathrm{d}\,t} = -b\,u + Q\,(t) \end{cases}, \tag{3}$$

where $P\,(t)\,, Q\,(t)$ are functions that describes the reinforcement rates. Japanese troops couldn't receive reinforcements so $Q\,(t) = 0$, see [4],[6]. The initial conditions are

$$\begin{cases} u\,(0) = u_{1,0} = 0 \\ v\,(0) = v_{2,0} = 21500 \end{cases}, \tag{4}$$

The last condition is the number of Japanese troop on the island. The solution, see [4], is

$$\begin{cases} u\,(t) = \sqrt{a/b}\,v_{2,0}\,\cosh\,\sqrt{a\,b}\,t - \int_0^t \cosh\,\sqrt{a\,b}\,(t-s) P\,(s)\,\mathrm{d}\,s \\ v\,(t) = v_{2,0}\,\cosh\,\sqrt{a\,b}\,t - \sqrt{a/b}\int_0^t \cosh\,\sqrt{a\,b}\,(t-s) P\,(s)\,\mathrm{d}\,s \end{cases}, \tag{5}$$

The calculus of the constants a, b is quite simple and we get also the results. It's possible to see the procedure consulting the cited Braun's book. Let's indicate with $x\,(\mu)$ the number of American troops, on the island, the μ-day of the battle.We get $b = 21500/\int_0^{36} x(t)\,\mathrm{d}t = 21500/2037000 = 0,0106$, $a = 20265/372500 = 0,054$. In the last equation we used the approximation

$$\int_0^{36} x(t)\,\mathrm{d}t \sim \sum_{h=0}^{36} x(h)$$

3 Critics of Lanchester Model. Importance of non linear model

First of all we observe that nobody has given a right prediction using Lanchester model. The Engel analysis of Iwo Jima battle was done after the fighting. Is an historical fact that Robert Strange McNamara[9] and his coworkers used Lanchesterś equations to try to understand events during the Vietnam war but, notwithstanding their efforts, they didn't succeed to get right prevision. We observe that Lanchester model is described by a couple of ODE. ODE are not suitable to describe real situations so there are many efforts to describe a *non linear* version of Lanchester's equations.[10] One attempt is to add noise terms to this equations, for example white noise a stochastic process with fixed mean and autocorrelation. So we get a couple of SDE[11].

$$\begin{cases} \frac{\mathrm{d}\,u}{\mathrm{d}\,t} = a\,v + \sum_{h=1}^{\infty} \hat{\nu}_i^2 \\ \frac{\mathrm{d}\,v}{\mathrm{d}\,t} = b\,u + \sum_{h=1}^{\infty} \hat{\omega}_i^2 \end{cases}, \tag{6}$$

The terms $\hat{\omega}_i\,, \hat{\nu}_i$ are the so called noise terms.They are assumed white with normal probability distribution $p(\omega) \sim \mathcal{N}(0, Q)$.[12] Solution of this set of SDE of this type isn't an easy job. But the results obtained are of interest in simulation combat.

As we said at the beginning is possible to represent troops on the terrain with a graph. We saw that troops on the terrain are example of a ACS. Given a network (or graph) it's possible to define the corresponding *adjacency* matrix A whose elements $g_{i,j}$ are (1). A related matrix is the *Laplacian* which takes values -1, for pairs of connected vertex and k_i degree of the corresponding node i in diagonal sites.

3.1 Example of non linear model-The Hirshleifer model

Adopting the precedents notations this model proposed by Hirshleifer, see [8], is described by a set of ODE of the type

$$\begin{cases} \frac{\mathrm{d}\,u}{\mathrm{d}\,t} = a\,u^{1-\lambda}\,v^{\lambda} \\ \frac{\mathrm{d}\,v}{\mathrm{d}\,t} = b\,v^{1-\lambda}\,u^{\lambda} \end{cases}, \tag{7}$$

[9] R.S McNamara (June 9, 1916 July 6, 2009) was an American business executive and the eighth Secretary of Defense, serving under Presidents John F. Kennedy and Lyndon B. Johnson from 1961 to 1968, during this time he played a large role in escalating the United States involvement in the Vietnam War

[10] We don't enter the definition of non linear system but the interested reader could find a throughly introduction in the book [5]

[11] Stochastic Differential Equations. SDE theory requires familiarity with advanced probability and stochastic processes

[12] In the appendix we recall the definition of a Gaussian measure

$\lambda \in [0,1]$ is a percentage. For different value of λ we get many interesting cases.

1. Case $\lambda = 0$. In this case the system (7) became

$$\begin{cases} \frac{\mathrm{d}\,u}{\mathrm{d}\,t} = a\,u \\ \frac{\mathrm{d}\,v}{\mathrm{d}\,t} = b\,v \end{cases},$$

   The solution are quite simple $u = C\,e^{a\,t}$ while $v = C\,e^{b\,t}$. We have indicate with C the constant of integration. In the plane In this case there is no combat. In fact in the plane (u, v) we get two exponential curves that evolving independently

2. Case $\lambda = 1/2$. In this case the system (7) became

$$\begin{cases} \frac{\mathrm{d}\,u}{\mathrm{d}\,t} = a\,\sqrt{u\,v} \\ \frac{\mathrm{d}\,v}{\mathrm{d}\,t} = b\,\sqrt{u\,v} \end{cases},$$

   In this case $\frac{\mathrm{d}\,u}{\mathrm{d}\,v} = \frac{a}{b}$. The solution is in this case very simple. If we denote with K a constant we get $b\,u - a\,v = K$. This equations describe a right line in the plane (u, v). In this situation is simple to analyze the values $a > b\,, a < b$ that describe the vantage conditions of a fighter respect the other. This is the case of the symmetric conflict

3. Case $\lambda = 1$. In this case the system (7) became

$$\begin{cases} \frac{\mathrm{d}\,u}{\mathrm{d}\,t} = a\,v \\ \frac{\mathrm{d}\,v}{\mathrm{d}\,t} = b\,u \end{cases},$$

   In this case $\frac{\mathrm{d}\,u}{\mathrm{d}\,v} = \frac{a\,v}{b\,u}$ and after integration we get, in the plane (u, v) the equation of a conic.We have relationships, formally, analogous to *The quadratic Lanchester Law*. In see [4]

It's interesting observe that $\frac{\mathrm{d}^2\,u}{\mathrm{d}\,t^2} - a\,b\,u = 0$. Posing $\omega = \sqrt{a\,b}$ we get, after integration $u = C\,\sin\left(\omega\,t + \varphi\right)$

3.2 Bifurcation methods

Bifurcation theory is a rather difficult subject so we don't wont to enter the details of this theory. We bound ourselves to the principal definition. First, trading to give a non linear version of the Lanchester, we observe that the function u($\cdot$) and v($\cdot$) of (2) must be regarded as functions of space and time. In particular the evolution problem is described by a PDE[13]. Let Ω indicate the square:

$$\Omega = [0,a] \times [0,a] = \{(x,y) \in \mathbf{R}^2 \mid 0 \le x \le l, 0 \le y \le l\} \subseteq \mathbf{R}^2$$

With this definition we can built the functional space of square integrable functions

$$L^2(\Omega) = \Big\{ u \in C^\infty(\Omega) \mid (\int_\Omega |u|^2 \,\mathrm{d}x\,\mathrm{d}y)^{\frac{1}{2}} < \infty \Big\}.$$

where $\mathrm{d}x\,\mathrm{d}y$ is the measure of Ω. Now we indicate with E a function spaces and consider $\mathrm{G} : \mathbf{R} \times \mathrm{E} \longrightarrow \mathrm{E}$. Let u$\in$E we wont to solve the equation

$$u = G(u, \lambda) \tag{8}$$

where $\lambda \in \mathbf{R}$. A very important question rises *it's possible to generate solution of (8) from a small perturbation of u?.* This is the central question of bifurcation theory.
We applied these ideas to this non linear version of Lanchester model.

$$\begin{cases} \frac{\partial u}{\partial t} = -a\,u + b\,u^\alpha\,v^\beta + \nabla^2 u \\ \frac{\partial v}{\partial t} = c\,v - d\,v^\beta\,u^\alpha + \nabla^2 v \end{cases}, \tag{9}$$

To apply bifurcation theory to (9) we introduce the functional space with compact support $C^2_0(\Omega) = \{u \in C^2(\Omega) \mid u(0) = u(a) = 0\}$. Let E$=C^2_0(\Omega)^2$ and F$=C^2(\Omega)^2$. We need to determine the stationary points P_0 that are solution of

$$\begin{cases} -a\,u + b\,u^\alpha\,v^\beta = 0 \\ c\,v - d\,v^\beta\,u^\alpha = 0 \end{cases},$$

At this point we consider a perturbation of this state, that mean to sum terms to the points in exam,

$$\begin{cases} u = u' + X^\beta\,Y^{1-\beta} \\ v = v' + Y^\alpha\,X^{1-\alpha} \end{cases},$$

where X, Y are functions of $a, b, c, d, \alpha + \beta$ Near the stationary point the system (9) became

$$\frac{\partial}{\partial t}\begin{pmatrix} u' \\ v' \end{pmatrix} = \mathcal{L}\begin{pmatrix} u' \\ v' \end{pmatrix} \tag{10}$$

where $\mathcal{L} : \mathrm{E} \times \mathbf{R} \longrightarrow \mathrm{F}$ and is the so called parabolic operator

$$\mathcal{L} = \begin{pmatrix} -a + \alpha\,b\,c + \nabla^2 & \beta\,b/b'\,a' \\ -\alpha\,b'/b\,a & a' - \beta\,b'\,c' + \nabla^2 \end{pmatrix}$$

[13] Partial differential equation

Bifurcation points are determined by the eigenvalue of $\mathcal{L}$. The solution of (10) is given by the Fourier series

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \sum_h e^{\nu_h\, t} \begin{pmatrix} u'_h \\ v'_h \end{pmatrix}.$$

If we put this equation in (10) we get

$$\mathcal{L}\begin{pmatrix} u'_j \\ v'_h \end{pmatrix} = \sum_h e^{\nu_h\, t} \begin{pmatrix} u'_h \\ v'_h \end{pmatrix}.$$

The stationary point P_0 is asymptotically stable if for all $m \in \mathbf{N}$ is verified the condition $\Re\,\nu_m < 0$ (Lyapunov theorem If $\Re\,\nu_m > 0$ the stationary point is asymptotically instable. For $\Re\,\nu_m = 0$ the point is a bifurcation point only if the eigenvalue has even multiplicity (Leray-Schauder theorem) If we consider only the unidimensional case $\nabla^2 = \frac{d}{d\,x}$ (in this case the problem is a Dirichlet problem)and assuming the eigenvalue are:

$$\begin{pmatrix} u_h \\ v_h \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.\ \sin \frac{m\,\pi\,x}{l}.$$

we get

$$\det \begin{pmatrix} -a(1-\alpha) - \nu_h - \frac{h^2\,\pi^2}{l^2} & \beta\, b/b'\, a' \\ -\alpha\, b'/b\, a & a'(1-\beta) - \nu_h - \frac{h^2\,\pi^2}{l^2} \end{pmatrix} \begin{pmatrix} u'_j \\ v'_h \end{pmatrix} = 0$$

Posing $\Delta_m = a(1-\alpha) - \frac{h^2\,\pi^2}{l^2}$ $\quad \Gamma_m = a'(1-\alpha) - \frac{h^2\,\pi^2}{l^2}$ we have to study the equation

$$\nu_m^2 + \nu_m(\Delta_m - \Gamma_m) - \Delta_m\,\Gamma_m + \alpha\,\beta\,a\,a' = 0. \tag{11}$$

Without entering a detailed discussion of the results we observe that the equation (11) is the equation of a curve of the second degree. It's possible, via the Graphic Software MATHEMATICA[14], to draw the graphic of equation (11) and recognize the regions $\nu_m < 0, \nu_m > 0$. These regions define, according the Lyapunov Theorem, the conditions of advantage of a fighter respect to the other.

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I'm indebted with my colleague and friend Maj.M. Di Tria who, carefully, reeded this notes and suggested me many things.

[14] This software is licensed by WOLFRAMRESEARCH

4 Appendix: Definition of Gaussian measure

Let $(\Omega, \mathcal{E}, \mu)$ a probability space and $(G, \mathcal{G})$ a measurable space. A random variable F in Ω with values in G is a mapping $F : \Omega \longrightarrow \Omega$ such that $F^{-1}(G) \in \mathcal{E}\,, \forall G \in \mathcal{G}$. When G is a topological space we denote by $\mathcal{B}(G)$ the σ-algebra of all Borel subset in G. Now let $a \in \mathcal{H}$ with inner product $\langle\cdot\rangle$ and norm $|\cdot|$. Let Q a linear operator $Q : \mathcal{H} \longrightarrow \mathcal{H}$ such that

$$\langle Q\,x, y\rangle = \langle x, Q\,y\rangle \quad \langle Q\,x, x\rangle \geq 0 \qquad \forall x, y \in \mathcal{H}.$$

In this case exists an orthonormal basis $\{e_1, e_2 \cdots e_k\}$ in $\mathcal{H}$ and non negative numbers $\{\lambda_1, \lambda_2 \cdots \lambda_k\}$ such that $Q\,e_k = \lambda_k\,e_k$. For any $x \in \mathcal{H}$ we set $x_k = \langle x, e_k\rangle$ and denote $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_k$ the Lebesgue measure $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$.

Definition 2 (Gaussian measure)
If $\det Q > 0$ *the Gaussian measure* $\mathcal{N}(a, Q)$ *on* $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ *is*

$$\mathcal{N}(a, Q) = \frac{1}{\sqrt{(2\,\pi)^n\,\det Q}} e^{-1/2\langle Q^{-1}(x-a), x-a\rangle}\,\mathrm{d}x$$

If $Q = 0 \quad \mathcal{N}(a, Q) = \delta_a$ *where* δ_a *is the Dirac measure concentrated in* a*. We recall that* $\int_{\mathbf{R}} f(x)\delta_a(x)\,\mathrm{d}x = f(a)$

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